A criterion for existence of solutions to the supercritical Bahri-Coron's problem

Saima Khenissy, Olivier Rey

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Saïma KHENISSY* and Olivier REY†

Abstract

We consider the supercritical elliptic problem $-\Delta u = u^{5+\varepsilon}$, $u > 0$ in Ω; $u = 0$ on $\partial\Omega$ with Ω a smooth bounded domain in $\mathbb{R}^3$, and $\varepsilon > 0$ a small number. Denoting by G the Green's function of $-\Delta$ on Ω with Dirichlet boundary conditions, and by H its regular part, we show that a nontrivial relative homology between the level sets φ^b and φ^a of φ, $a < b < 0$, $\varphi(x,y) = H(x,x)^{1/2}H(y,y)^{1/2} - G(x,y)$, implies the existence, for ε small enough, of a solution to the problem which blows up, as ε goes to 0, at two points x, y such that $a \leq \varphi(x,y) \leq b$, $\nabla\varphi(x,y) = 0$.

1 Introduction

Let us consider the nonlinear elliptic problem

$$(P_q) \qquad \begin{cases} -\Delta u &= u^q \quad , u > 0 \qquad \text{in } \Omega \\ \quad u &= 0 \qquad\qquad\qquad \text{on } \partial\Omega \end{cases}$$

where $1 < q < +\infty$, and Ω is a smooth and bounded domain in $\mathbb{R}^N$, $N \geq 3$.

When q is subcritical, i.e. $q < \frac{N+2}{N-2}$, the mountain pass lemma proves the existence of a solution to (P_q) for any domain Ω. In the case $q = \frac{N+2}{N-2}$, Pohozaev's identity [10] shows that problem (P_q) has no solution as Ω is starshaped. On the other hand Kazdan and Warner [6] proved that a solution exists in the special case Ω is an annulus, and Bahri and Coron [1] showed that a nontrivial topology of the domain, in the sense $H_{2n-1}(\Omega;\mathbb{Q}) \neq 0$ or $H_n(\Omega;\mathbb{Z}/2\mathbb{Z}) \neq 0$ for some $n \in \mathbb{N}^*$, implies that (P_q) has a solution. When $q > \frac{N+2}{N-2}$, Passaseo [8, 9] gives, for $N \geq 4$, an example of a topologically nontrivial domain for which no solution of (P_q) exists, provided that $q > \frac{N+1}{N-3}$. The present paper is interested in the slightly supercritical case, i.e. $q = \frac{N+2}{N-2} + \varepsilon$, with $\varepsilon > 0$ a small number. More precisely, we consider, for $\xi \in \mathbb{R}^N$ and $\lambda > 0$, the functions

$$\overline{U}_{\xi,\lambda}(x) = \alpha_N \left(\frac{\lambda}{\lambda^2 + |x-\xi|^2} \right)^{\frac{N-2}{2}}, \quad x \in \mathbb{R}^N$$

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with $\alpha_N = \big[N(N-2)\big]^{\frac{N-2}{4}}$. These functions are the only solutions to the equation

$$-\Delta u = u^p, \quad u > 0 \qquad \text{in} \quad \mathbb{R}^N$$

with $p = \frac{N+2}{N-2}$ ([4]). We denote by $U_{\xi,\lambda}$ the projection onto $H_0^1(\Omega)$ of $\overline{U}_{\xi,\lambda}$, defined by

$$\begin{cases} \Delta U_{\xi,\lambda} &= \Delta \overline{U}_{\xi,\lambda} \qquad \text{in } \Omega \\ U_{\xi,\lambda} &= 0 \qquad \text{on } \partial\Omega. \end{cases}$$

Writing $U_{\xi,\lambda} = \overline{U}_{\xi,\lambda} - f_{\xi,\lambda}$, $f_{\xi,\lambda}$ solves

$$\begin{cases} -\Delta f_{\xi,\lambda} &= 0 \qquad \text{in } \Omega \\ f_{\xi,\lambda} &= \overline{U}_{\xi,\lambda} \qquad \text{on } \partial\Omega. \end{cases}$$

Functions $U_{\xi,\lambda}$ are approximate solutions to (P_p) when $\xi \in \Omega$ and λ goes to zero. This fact suggests looking for solutions to (P_q), q close to p, in a neighbourhood of the $U_{\xi,\lambda}$'s, that is for solutions writing as

$$u = \sum_{i=1}^{k} U_{\xi_i,\lambda_i} + v$$

with $k \in \mathbb{N}^*$ and v small in some norm.

When $q = p - \varepsilon$, Bahri, Li and Rey [2], Rey [12] proved that such solutions exist. In particular, a solution exists for $k = 1$ and ε small enough, which blows up at a critical point of the Robin function $H(x,x)$ as ε goes to 0.

When $q = p + \varepsilon$, the situation comes out to be different. Ben Ayed, Grossi, El Mehdi and Rey [3] showed that there is no solution blowing up at a single point as ε goes to 0. However, Del Pino, Felmer and Musso [5] proved, under certain topological conditions, the existence of such solutions when $k = 2$. To state their result, some notations have to be introduced.

We denote by G the Green's function of the operator $(-\Delta)$ with Dirichlet boundary conditions on Ω, by H its regular part, i.e.

$$G(x,y) = b_N \frac{1}{|x-y|^{N-2}} - H(x,y)$$

with $b_N = \big[(N-2)\sigma_{N-1}\big]^{-1}$, σ_{N-1} is the measure of the $(N-1)$-dimensional unit sphere, and

$$\begin{cases} \Delta_x H &= 0 \qquad \text{in } \Omega \times \Omega \\ H(x,y) &= b_N \frac{1}{|x-y|^{N-2}} \qquad \text{on } \partial(\Omega \times \Omega) \end{cases}$$

For $(\xi_1, \xi_2) \in \Omega \times \Omega$ we define

$$\varphi(\xi_1,\xi_2) = H^{\frac{1}{2}}(\xi_1,\xi_1) H^{\frac{1}{2}}(\xi_2,\xi_2) - G(\xi_1,\xi_2). \tag{1.1}$$

$H^d(B)$ denotes the d-th cohomology group with integral coefficients of $B \subset \Omega$, and $i^* : H^*(\Omega) \to H^*(B)$ the homomorphism induced by the inclusion $i : B \to \Omega$. We have :

Theorem[5]. *Assume $3 \leq N \leq 6$ and let Ω be a bounded domain with smooth boundary in $\mathbb{R}^N$. Suppose there exists a compact manifold $\mathcal{M} \subset \Omega$ and an integer $d \geq 1$ such that $\varphi < 0$ on $\mathcal{M} \times \mathcal{M}$, $i^* : H^d(\Omega) \to H^d(\mathcal{M})$ is nontrivial, and either d is odd or $H^{2d}(\Omega) = 0$.*

Then there exists $\varepsilon_0 > 0$ such that, for any $0 < \varepsilon < \varepsilon_0$, problem $(P_{p+\varepsilon})$ has at least one solution u_ε. Moreover, let $\mathcal{C}$ be the component of the set where $\varphi < 0$ which contains $\mathcal{M} \times \mathcal{M}$. Then, given any sequence (ε_n) going to zero, there is a subsequence, which we denote in the same way, and a critical point $(\xi_1, \xi_2) \in \mathcal{C}$ of the function φ such that $u_{\varepsilon_n}(x) \to 0$ on compact subsets of $\Omega \setminus \{\xi_1, \xi_2\}$ and such that for any $\delta > 0$

$$\sup_{|x-\xi_i|<\delta} u_{\varepsilon_n}(x) \to +\infty, \quad \textit{as } \varepsilon \to 0, \quad i = 1, 2.$$

Examples are given of domains satisfying the topological assumptions of the theorem. Namely, a fixed domain $\mathcal{D}$ in $\mathbb{R}^N$, from which a subdomain ω is excised, ω contained in a ball of sufficiently small radius. When $N = 3$, another example consists of an arbitrary domain $\mathcal{D}$, from which a solid torus is excised, with sufficiently small cross-section.

In the present work, we consider the case $N = 3$. We prove again the existence of the two-bubble solutions to $(P_{5+\varepsilon})$, blowing up at two points as ε goes to 0, with some modifications of the method used in [5], leading to a simpler and more natural assumption. Indeed, denoting by

$$\varphi^a = \{x \in \Omega \times \Omega \,/\, \varphi(x) \le a\}$$

the level sets of φ, we prove :

Theorem 1.1 *$N = 3$. Assume that $a < b < 0$, b is not a critical value of φ, and the relative homology $H_*(\varphi^b, \varphi^a)$ is nontrivial. Then, for ε small enough, there exists a sequence of solutions to $(P_{5+\varepsilon})$ which blows up at ξ_1, ξ_2 as ε goes to zero, with (ξ_1, ξ_2) a critical point of φ such that $a < \varphi(\xi_1, \xi_2) < b$.*

Furthermore, denoting by

$$\Delta = \{(x_1, x_2) \in \Omega \times \Omega \,/\, x_1 = x_2\}$$

the diagonal of $\Omega \times \Omega$, we have :

Corollary 1.1 *$N = 3$. Assume that $b \le 0$ is not a critical value of φ and $H_*(\varphi^b, \Delta) \neq 0$. Then, the problem $(P_{5+\varepsilon})$ has a solution as described in* Theorem 1.1*, with $\varphi(\xi_1, \xi_2) < b$.*

The corollary follows from the fact that φ^a retracts by deformation on Δ, for a small enough, as proved at the end of the paper.

Remarks.
1) The arguments are still valid for $N = 4$. However, for sake of simplicity, we restrict ourselves in this paper to the case $N = 3$. Dimensions $N \ge 5$ could be treated using the same tools as in [13].

2) The problem has a variational structure. Let J_q denote a functional whose critical points are solutions to $(P_q$. For slightly subcritical exponents, i.e. $q = \frac{N+2}{N-2} - \varepsilon$, the difference of topology induced by the solutions blowing up at two points, as ε goes to zero, between the level sets of J_q, is linked to the relative topology between Ω^2 and $\varphi^- = \{x \in \Omega \times \Omega \,/\, \varphi(x) \le 0\}$ [2]. Here, the relevant quantity is the relative topology between φ^- and Δ. We know that for expanding annuli-domains A_η, $\eta < |x| < \eta^{-1}$, φ^- is homotopically equivalent to $A_\eta \times A_\eta$, whereas for thin annuli-domains φ^- is homotopically equivalent to Δ [7]. In one case, we obtain blowing up

solutions for the supercritical exponent going to the critical one, in the other case blowing up solutions for the subcritical exponent going to the critical one.

Our approach to prove Theorem 1.1 is similar to Del Pino, Felmer and Musso's one, with appropriate modifications. In Section 2, we recall some results of [5] which lead to a finite-dimensional reduction of the problem, together with an asymptotic expansion of the reduced functional. Section 3 is devoted to the additional results which are required in Section 4 to prove Theorem 1.1.

2 Known results : the finite dimensional reduction

We consider the energy functional, defined in $H_0^1(\Omega) \cap L^{6+\varepsilon}(\Omega)$, whose positive critical points are solutions to the problem $(P_\varepsilon) := (P_{5+\varepsilon})$

$$J_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{6+\varepsilon}\int_\Omega u_+^{6+\varepsilon}.$$

We look for solutions to (P_ε) in a neighbourhood of functions $U = U_1 + U_2$, $U_i = U_{\xi_i,\lambda_i}$ denoting the $H_0^1(\Omega)$-projection of $\overline{U}_i = \overline{U}_{\xi_i,\lambda_i}$ introduced in Section 1. Let $\delta > 0$ be a small number, we set

$$\mathcal{O}_\delta(\Omega) = \big\{(\xi_1,\xi_2) \in \Omega\times\Omega\ : |\xi_1 - \xi_2| > \delta,\ dist(\xi_i, \partial\Omega) > \delta,\ i = 1,2\big\}$$

and

$$\lambda_i = c\Lambda_i^2\varepsilon, \quad i = 1,2, \qquad\qquad \delta < \Lambda_i < \delta^{-1} \tag{2.1}$$

with

$$c = \frac{b_3\int_{\mathbb{R}^3}\overline{U}^6}{6\alpha_3\int_{\mathbb{R}^3}\overline{U}^5} = \frac{1}{128} \qquad\qquad \overline{U} = \overline{U}_{0,1}.$$

We also define

$$\Psi(\xi_1,\xi_2,\Lambda_1,\Lambda_2) = \frac{1}{2}\Big(H(\xi_1,\xi_1)\Lambda_1^2 + H(\xi_2,\xi_2)\Lambda_2^2 - 2G(\xi_1,\xi_2)\Lambda_1\Lambda_2\Big) + \ln\Lambda_1\Lambda_2. \tag{2.2}$$

Expanding $J_\varepsilon(U_1 + U_2)$ with respect to ε, we find ([2], [5]) :

Lemma 2.1 *There exist constants $C_1 > 0$, $C_2 > 0$, C_3 such that, for any $\delta > 0$*

$$J_\varepsilon(U_1+U_2) = C_1 + C_2\varepsilon\ln\varepsilon + C_3\varepsilon + C_2\varepsilon\Psi(\xi_1,\xi_2,\Lambda_1,\Lambda_2) + o(\varepsilon)$$

uniformly with respect to $(\xi_1,\xi_2,\Lambda_1,\Lambda_2) \in \mathcal{O}_\delta(\Omega)\times]\delta,\delta^{-1}[^2$. This expansion holds in C^2-norm with respect to the variables ξ and Λ in the considered domain.

Remark. Writing $\overline{U} = \overline{U}_{0,1}$, constants C_1, C_2 and C_3 are defined as

$$\begin{aligned} C_1 &= 2\frac{1}{3}\int_{\mathbb{R}^3}\overline{U}^6 = \sqrt{3}\frac{\pi^2}{2} \\ C_2 &= \frac{1}{6}\int_{\mathbb{R}^3}\overline{U}^6 = \sqrt{3}\frac{\pi^2}{8} \\ C_3 &= \frac{1}{6}\left(\frac{1}{3} + \ln c\right)\int_{\mathbb{R}^3}\overline{U}^6 - \frac{1}{3}\int_{\mathbb{R}^3}\overline{U}^6\ln\overline{U}. \end{aligned} \tag{2.3}$$

In view of this result, the strategy to prove the theorem consists in reducing the original problem of finding a critical point of J_ε to a finite dimensional one in the variables ξ_1, ξ_2, Λ_1, Λ_2, such that the critical points of Ψ will provide us with the solutions that we are looking for. We first perform a rescaling. We set

$$\Omega_\varepsilon = \Omega/\varepsilon$$

and $\delta > 0$ being fixed, we consider points $\xi_i^{'} \in \Omega_\varepsilon$, numbers $\Lambda_i > 0$, $i = 1, 2$, such that

$$|\xi_1^{'} - \xi_2^{'}| > \frac{\delta}{\varepsilon}, \quad \mathrm{dist}(\xi_i^{'}, \partial\Omega_\varepsilon) > \frac{d}{\varepsilon}, \quad \delta < \Lambda_i < \delta^{-1}. \tag{2.4}$$

We define the functions

$$\overline{V}_i = \overline{U}_{\xi_i^{'}, \Lambda_i^*} = \alpha_N \Big(\frac{\Lambda_i^*}{(\Lambda_i^*)^2 + |\cdot - \xi_i^{'}|^2} \Big)^{\frac{1}{2}} \qquad \Lambda_i^* = c\Lambda_i^2.$$

As previously, we define the projections onto $H_0^1(\Omega_\varepsilon)$ of these functions, namely the functions V_i given as the unique solutions of

$$\begin{cases} -\Delta V_i & = \overline{V}_i^5 \quad \text{in} \quad \Omega_\varepsilon \\ \quad V_i & = 0 \quad \text{on} \quad \partial\Omega_\varepsilon. \end{cases}$$

Let us denote

$$V = V_1 + V_2 \qquad \overline{V} = \overline{V}_1 + \overline{V}_2 \qquad \text{and} \qquad (\xi_1^{'}, \xi_2^{'}, \Lambda_1, \Lambda_2) = (\xi^{'}, \Lambda).$$

Our aim is finding a solution to the problem in a neighbourhood of V, for appropriate $\xi^{'}$ and Λ. In order to reduce the problem to a finite dimensional one, we first solve the linearized equation at V. We define, for $i = 1, 2$, the functions

$$\overline{Z}_{ij} = \frac{\partial \overline{V}_i}{\partial \xi_{ij}^{'}} \qquad 1 \le j \le 3 \qquad \overline{Z}_{i4} = \frac{\partial \overline{V}_i}{\partial \Lambda_i^*}$$

which span the kernel of the linearized problem at $\overline{V}_i$ on $\mathbb{R}^3$ when $\varepsilon = 0$, and their $H_0^1(\Omega_\varepsilon)$-projections Z_{ij}, i.e. the unique solutions of

$$\begin{cases} \Delta Z_{ij} & = \Delta \overline{Z}_{ij} \quad \text{in} \quad \Omega_\varepsilon \\ \quad Z_{ij} & = 0 \quad \text{on} \quad \partial\Omega_\varepsilon. \end{cases}$$

Then, following [5], we consider the problem : $h \in L^\infty(\Omega_\varepsilon)$ being given, find a function ϕ which satisfies

$$\begin{cases} \Delta\phi + (5+\varepsilon)V^{4+\varepsilon}\phi & = h + \sum_{i,j} c_{ij} V_i^4 Z_{ij} \quad \text{in} \quad \Omega_\varepsilon \\ \phi & = 0 \quad \text{on} \quad \partial\Omega_\varepsilon \\ < V_i^4 Z_{ij}, \phi > & = 0 \quad \text{for all} \quad i, j \end{cases} \tag{2.5}$$

for some numbers $c_{i,j}$, $< \cdot, \cdot >$ denoting the scalar product in $L^2(\Omega_\varepsilon)$ (we notice that the orthogonality to $V_i^4 Z_{ij}$ in $L^2(\Omega_\varepsilon)$ is equivalent to the orthogonality to Z_{ij} in $H_0^1(\Omega_\varepsilon)$).

Existence and uniqueness of ϕ follows from the implicit functions theorem, in suitable functional spaces. Actually, for ψ a function defined on Ω_ε, we consider the following weighted L^∞-norms

$$\|\psi\|_* = \sup_{x\in\Omega_\varepsilon}\Big|\Big((1+|x-\xi_1^{'}|^2)^{-\frac{1}{2}} + (1+|x-\xi_2^{'}|^2)^{-\frac{1}{2}}\Big)^{-1}\psi(x)\Big|$$

and

$$\|\psi\|_{**} = \sup_{x\in\Omega_\varepsilon}\Big|\Big((1+|x-\xi_1^{'}|^2)^{-\frac{1}{2}} + (1+|x-\xi_2^{'}|^2)^{-\frac{1}{2}}\Big)^{-4}\psi(x)\Big|.$$

These norms are equivalent to $\|(\overline{V})^{-1}\psi\|_\infty$ and $\|(\overline{V})^{-4}\psi\|_\infty$ respectively, uniformly with respect to the points and numbers satisfying (2.4).

We recall the following result (see Propositions 4.1 and 4.2 in [5]) :

Proposition 2.1 *Assume that conditions* (2.4) *hold. There exists $\varepsilon_0 > 0$ and a constant $C > 0$ independent of ε, ξ', Λ, such that for all $0 < \varepsilon < \varepsilon_0$ and all $h \in L^\infty(\Omega_\varepsilon)$, problem* (2.5) *has a unique solution $\phi \equiv L_\varepsilon(h)$, which satisfies*

$$\|L_\varepsilon(h)\|_* \leq C\|h\|_{**} \qquad\qquad |c_{ij}| \leq C\|h\|_{**}. \tag{2.6}$$

Moreover, the map $(\xi^{'},\Lambda) \mapsto L_\varepsilon(h)$ is C^1 and

$$\|D_{(\xi^{'},\Lambda)}\,L_\varepsilon(h)\|_* \leq C\|h\|_{**}. \tag{2.7}$$

A first order correction to consider, between V and a solution to the original problem, is given by

$$\psi = -L_\varepsilon(R^\varepsilon) \qquad\qquad R^\varepsilon = V^{5+\varepsilon} - \overline{V}_1^5 - \overline{V}_2^5. \tag{2.8}$$

Estimating $\|R^\varepsilon\|_{**}$ - see 3.22 below - provides us, through Proposition 2.1, with an estimate of $\|\psi\|_*$. Namely, there exists a constant C, independent of ε and ξ', Λ satisfying (2.4) such that

$$\|\psi\|_* \leq C\varepsilon. \tag{2.9}$$

Still following [5], the next step is considering the nonlinear problem of finding ϕ such that, for some numbers c_{ij}, the following holds

$$\left\{\begin{array}{rll} \Delta(V+\psi+\phi) + (V+\psi+\phi)_+^{5+\varepsilon} & = \sum_{i,j} c_{ij}V_i^4 Z_{ij} & \text{in} \quad \Omega_\varepsilon \\ \phi & = 0 & \text{on} \quad \partial\Omega_\varepsilon \\ < V_i^4 Z_{ij}, \phi > & = 0 & \text{for all} \quad i,j. \end{array}\right. \tag{2.10}$$

Setting

$$N_\varepsilon(\eta) = (V+\eta)_+^{5+\varepsilon} - V^{5+\varepsilon} - (5+\varepsilon)V^{4+\varepsilon}\eta \tag{2.11}$$

system (2.10) writes as

$$\left\{\begin{array}{rll} \Delta\phi + (5+\varepsilon)V^{4+\varepsilon}\phi & = -N_\varepsilon(\psi+\phi) + \sum_{i,j} c_{ij}V_i^4 Z_{ij} & \text{in} \quad \Omega_\varepsilon \\ \phi & = 0 & \text{on} \quad \partial\Omega_\varepsilon \\ < V_i^4 Z_{ij}, \phi > & = 0 & \text{for all} \quad i,j. \end{array}\right. \tag{2.12}$$

Lemma 5.1 in [5] provides us with the following estimate, when conditions (2.4) hold : there is a positive constant C such that, for any sufficiently small ε and $\|\phi\|_* \leq 1$

$$\|N_\varepsilon(\psi+\phi)\|_{**} \leq C(\|\phi\|_*^2 + \varepsilon^2). \tag{2.13}$$

Then, applying a fixed point theorem to the map A_ε from $\mathcal{F} = \{\phi \in H_0^1 \cap L^\infty(\Omega_\varepsilon) \ : \ \|\phi\|_* \leq \varepsilon\}$ to $H_0^1 \cap L^\infty(\Omega_\varepsilon)$ defined as

$$A_\varepsilon(\phi) = -L_\varepsilon(N_\varepsilon(\phi+\psi))$$

which is contracting in norm $\|\cdot\|_*$, we obtain (see Propositions 5.1 and 5.2 in [5]) :

Proposition 2.2 *Assume that conditions* (2.4) *hold. For ε small enough, there exists a unique solution $\phi = \phi(\xi', \Lambda)$ to problem* (2.12) *in $\mathcal{F}$. Moreover, $(\xi', \Lambda) \to \phi(\xi', \Lambda)$ is C^1, and there exists a constant $C > 0$ independent of ε, ξ', Λ, such that*

$$\|D_{(\xi',\Lambda)}\, \phi\|_* \leq C\varepsilon. \tag{2.14}$$

Remark. Actually, $\phi = \phi(\xi', \Lambda)$ and its first derivatives satisfy the estimates

$$\|\phi\|_* \leq C\varepsilon^2 \qquad\qquad \|D_{(\xi',\Lambda)}\, \phi\|_* \leq C\varepsilon^2. \tag{2.15}$$

The first inequality is a consequence of (2.13) and the definition of ϕ as a fixed point of A_ε. The second one is proved in the next section - see Lemma 3.3.

To come back to the original problem, we consider the rescaled functions defined in Ω

$$\hat{\psi}(\xi,\Lambda)(x) = \varepsilon^{-\zeta}\psi(\varepsilon^{-1}\xi,\Lambda)(\varepsilon^{-1}x) \qquad \zeta = \frac{1}{2+\frac{1}{2}\varepsilon} \tag{2.16}$$

$$\hat{\phi}(\xi,\Lambda)(x) = \varepsilon^{-\zeta}\phi(\varepsilon^{-1}\xi,\Lambda)(\varepsilon^{-1}x) \tag{2.17}$$

and

$$\hat{U}(\xi,\Lambda)(x) = \varepsilon^{-\zeta}V(\varepsilon^{-1}x) = \varepsilon^{(\frac{1}{2}-\zeta)}(U_1(x)+U_2(x)) \qquad\qquad U_i(x) = U_{\xi_i,\lambda_i}(x) \tag{2.18}$$

with

$$\lambda_i = c\Lambda_i^2\varepsilon \quad \text{and} \quad \xi_i = \varepsilon\xi_i' \in \mathcal{O}_\delta(\Omega).$$

Lastly, we define

$$I_\varepsilon(\xi,\Lambda) \equiv J_\varepsilon\big((\hat{U}+\hat{\psi}+\hat{\phi})(\xi,\Lambda)\big). \tag{2.19}$$

Previous results provide us with the following basic assertion (see Lemma 6.1 in [5]) :

Proposition 2.3 *The function $u = \hat{U}+\hat{\psi}+\hat{\phi}$ is a solution to problem (P_ε) if and only if (ξ,Λ) is a critical point of I_ε.*

Consequently we are led, for proving the theorem, to find a critical point of I_ε. We establish before, in the next section, some results about the second derivatives of I_ε.

3 Improved results : the second derivatives of I

In this section, we prove a C^2-expansion of I_ε - see (3.33) below. For this purpose, we first show that inequalities (2.7) and (2.14) are still valid for the second derivatives of L_ε and ϕ respectively.

Lemma 3.1 *Assume that assumptions of Proposition* 2.1 *are satisfied.* $L_\varepsilon(h)$ *is* C^2 *with respect to* $\Lambda, \xi^{'}$*, and there is a constant* C *independent of* ε*,* $\xi^{'}$*,* Λ*, such that*

$$\|D^2_{(\xi^{'},\Lambda)} L_\varepsilon(h)\|_* \le C\|h\|_{**}. \tag{3.1}$$

Proof. For given $h \in L^\infty(\Omega_\varepsilon)$, we recall that problem

$$\left\{\begin{array}{rll} \Delta\phi + (5+\varepsilon)V^{4+\varepsilon}\phi & = h + \sum_{i,j} c_{ij} V_i^4 Z_{ij} & \text{in} \quad \Omega_\varepsilon \\ \phi & = 0 & \text{on} \quad \partial\Omega_\varepsilon \\ < V_i^4 Z_{ij}, \phi > & = 0 & \text{for all} \quad i,j \end{array}\right. \tag{3.2}$$

has a unique solution $\phi = L_\varepsilon(h)$ for some numbers c_{ij}, with

$$\|L_\varepsilon(h)\|_* \le C\|h\|_{**} \qquad\qquad |c_{ij}| \le C\|h\|_{**}. \tag{3.3}$$

Let us write

$$Z = \partial_{\xi'}\phi$$

at least formally. Differentiating the first and third equations in (3.2) with respect to $\xi^{'}$, we find

$$\begin{aligned} \Delta Z + (5+\varepsilon)V^{4+\varepsilon} Z = & -(5+\varepsilon)(\partial_{\xi'} V^{4+\varepsilon})\phi \\ & + \sum_{i,j} d_{ij} V_i^4 Z_{ij} + \sum_{i,j} c_{ij}\partial_{\xi'}(V_i^4 Z_{ij}) \quad \text{in} \quad \Omega_\varepsilon \end{aligned} \tag{3.4}$$

with $d_{ij} = \partial_{\xi'} c_{ij}$ and

$$< \partial_{\xi'}(V_i^4 Z_{ij}), \phi > + < V_i^4 Z_{ij}, Z >= 0 \quad \text{for all} \quad i,j. \tag{3.5}$$

We decompose Z, writing

$$Z = \tilde{Z} + \sum_{lk} b_{lk} Z_{lk} \quad \text{with} \quad < V_i^4 Z_{ij}, \tilde{Z} >= 0 \quad \forall i,j.$$

It follows from (3.5) that

$$\sum_{lk} b_{lk} < V_i^4 Z_{ij}, Z_{lk} >= - < \partial_{\xi'}(V_i^4 Z_{ij}), \phi > \quad \forall i,j. \tag{3.6}$$

This defines a linear system in the b_{lk}'s which is almost diagonal, with uniformly bounded coefficients since, as ε goes to 0, we have

$$< V_i^4 Z_{ij}, Z_{lk} >= \delta_{i,l}\delta_{j,k} \int_{\mathbb{R}^N} \overline{U}^4_{0,\Lambda_i^*} \Big(\frac{\partial \overline{U}_{0,\Lambda_i^*}}{\partial y_j}\Big)^2 + o(1) \tag{3.7}$$

with $y_j = x_j$, $1 \leq j \leq 3$, and $y_4 = \Lambda_i^*$. Such a system is uniquely solvable, and estimating the right hand side provides us with $b_{lk} = O(\|\phi\|_*)$. Whence, using (3.3)

$$b_{lk} = O(\|h\|_{**}). \tag{3.8}$$

On the other hand, using (3.4) we find that $\tilde{Z}$ satisfies the equation

$$\Delta\tilde{Z} + (5+\varepsilon)V^{4+\varepsilon}\tilde{Z} = f + \sum_{i,j} d_{ij}V_i^4 Z_{ij} \quad \text{in} \quad \Omega_\varepsilon \tag{3.9}$$

with

$$f = -(5+\varepsilon)(\partial_{\xi'}V^{4+\varepsilon})\phi - \sum_{i,j} b_{ij}[\Delta Z_{ij} + (5+\varepsilon)V^{4+\varepsilon}Z_{ij}] + \sum_{i,j} c_{ij}\partial_{\xi'}(V_i^4 Z_{ij}). \tag{3.10}$$

It is easily checked, taking account of (3.3) and (3.8), that

$$\|f\|_{**} \leq C\|h\|_{**}.$$

Then, we deduce from Proposition 2.1 that $\tilde{Z} \equiv L_\varepsilon(f)$ and d_{ij} satisfy

$$\|\tilde{Z}\|_* \leq C\|h\|_{**} \qquad\qquad |d_{ij}| \leq C\|h\|_{**}. \tag{3.11}$$

Thus, using again estimate (3.8) we obtain

$$\|Z\|_* \leq C\|h\|_{**}. \tag{3.12}$$

Once these estimates are known, one easily check that

$$Z = L_\varepsilon(f) + + \sum_{lk} b_{lk}Z_{lk}$$

with the b_{lk}'s defined by (3.6) and f defined by (3.10) is definitely the derivative of ϕ with respect to $\xi^{'}$.

We have now to estimate the second derivatives of ϕ with respect to $\xi^{'}$. Differentiating (3.4) and (3.5) with respect to $\xi^{'}$, and writing formally

$$W = \partial_{\xi'}Z = \partial^2_{\xi'^2}\phi$$

we find

$$\begin{aligned} \Delta W + (5+\varepsilon)V^{4+\varepsilon}W &= -2(5+\varepsilon)(\partial_{\xi'}V^{4+\varepsilon})Z - (5+\varepsilon)(\partial^2_{\xi'}V^{4+\varepsilon})\phi \\ &\quad + \sum_{i,j}\left[e_{ij}V_i^4 Z_{ij} + 2d_{ij}\partial_{\xi'}(V_i^4 Z_{ij}) + c_{ij}\partial^2_{\xi'}(V_i^4 Z_{ij})\right] \end{aligned} \tag{3.13}$$

with $e_{ij} = \partial_{\xi'} d_{ij}$ and

$$< \partial^2_{\xi'^2}(V_i^4 Z_{ij}), \phi > +2 < \partial_{\xi'}(V_i^4 Z_{ij}), Z > + < V_i^4 Z_{ij}, W >= 0 \quad \forall i, j. \tag{3.14}$$

We proceed as previously, writing W as

$$W = \tilde{W} + \sum_{l,k} a_{lk}Z_{lk} \quad \text{with} \quad < V_i^4 Z_{ij}, \tilde{W} >= 0 \quad \forall i, j.$$

It follows from (3.14) that

$$\sum_{l,k} a_{lk} < V_i^4 Z_{ij}, Z_{lk} >= - < \partial^2_{\xi'^2}(V_i^4 Z_{ij}), \phi > -2 < \partial_{\xi'}(V_i^4 Z_{ij}), Z > \quad \forall i, j. \tag{3.15}$$

(3.7) implies again that (3.15) is a linear system in the a_{lk}'s which is almost diagonal, and (3.3), (3.12) show that

$$|a_{lk}| \le C(\|\phi\|_* + \|Z\|_*) \le C'\|h\|_{**}. \tag{3.16}$$

On the other hand, we have

$$\|(\partial_{\xi'} V^{4+\varepsilon})Z\|_{**} \le C\|Z\|_* \le C'\|h\|_{**} \qquad \|(\partial^2_{\xi'^2} V^{4+\varepsilon})\phi\|_{**} \le C\|\phi\|_* \le C'\|h\|_{**} \tag{3.17}$$

and

$$\|\partial_{\xi'}(V_i^4 Z_{ij})\|_{**} \le C \qquad \|\partial^2_{\xi'^2}(V_i^4 Z_{ij})\|_{**} \le C. \tag{3.18}$$

Going back to (3.13), we see that $\tilde{W}$ satisfies

$$\begin{cases} \Delta \tilde{W} + (5+\varepsilon)V^{4+\varepsilon}\tilde{W} & = g + \sum_{i,j} e_{ij} V_i^4 Z_{ij} \qquad \text{in} \quad \Omega_\varepsilon \\ \tilde{W} & = 0 \qquad \text{on} \quad \partial\Omega_\varepsilon \\ < V_i^4 Z_{ij}, \tilde{W} > & = 0 \qquad \text{for all} \quad i, j \end{cases} \tag{3.19}$$

with

$$\begin{aligned} g = & -\sum_{i,j} a_{ij}[\Delta Z_{ij} + (5+\varepsilon)V^{4+\varepsilon} Z_{ij}] + 2\sum_{i,j} d_{ij}\partial_{\xi'}(V_i^4 Z_{ij}) + \sum_{i,j} c_{ij}\partial^2_{\xi'^2}(V_i^4 Z_{ij}) \\ & - (5+\varepsilon)(\partial^2_{\xi'^2} V^{4+\varepsilon})\phi - 2(5+\varepsilon)(\partial_{\xi'} V^{4+\varepsilon})Z. \end{aligned} \tag{3.20}$$

Estimates (3.3), (3.11), (3.16), (3.17) and (3.18) prove that

$$\|g\|_{**} \le C\|h\|_{**}$$

whence, in view of Proposition 2.1

$$\|\tilde{W}\|_* \le C\|h\|_{**}.$$

Finally, estimate (3.16) yields

$$\|W\|_* \le C\|h\|_{**}.$$

Once again, these estimates show that $W = L_\varepsilon(g) + \sum_{l,k} a_{lk} Z_{lk}$, where the a_{lk}'s are defined by (3.15) and g by (3.20), is indeed the second derivative of ϕ with respect to $\xi^{'}$. The other first and second derivatives of ϕ, involving the Λ_i's, may be treated in the same way. This concludes the proof of Lemma 3.1. □

Lemma 3.2 *The map $(\xi^{'}, \Lambda) \mapsto \psi(\xi^{'}, \Lambda)$ is C^2 for the norm $\|\cdot\|_*$, and there exists C independent of ε and ξ', Λ satisfying (2.4) such that*

$$\|\psi\|_* \le C\varepsilon \qquad \|D_{(\xi',\Lambda)}\psi\|_* \le C\varepsilon \qquad \|D^2_{(\xi',\Lambda)}\psi\|_* \le C\varepsilon. \tag{3.21}$$

Proof. We recall that ψ is defined as

$$\psi = -L_\varepsilon(R^\varepsilon) \qquad R^\varepsilon = (V_1+V_2)^{5+\varepsilon} - \overline{V}_1^5 - \overline{V}_2^5.$$

The smoothness of $(\xi', \Lambda) \mapsto \psi(\xi', \Lambda)$ follows from the smoothness of R^ε with respect to (ξ', Λ) and Lemma 3.1. Moreover, in view of Proposition 2.1 and Lemma 3.1, it is sufficient for establishing (3.21) to prove

$$\|R^\varepsilon\|_{**} \le C\varepsilon \qquad \|D_{(\xi',\Lambda)}R^\varepsilon\|_{**} \le C\varepsilon \qquad \|D^2_{(\xi',\Lambda)}R^\varepsilon\|_{**} \le C\varepsilon \tag{3.22}$$

as ε goes to zero.

Let us estimate $\|R^\varepsilon\|_{**}$. In the regions $|x-\xi_i'| \le \overline{\delta}/\varepsilon$, for small $\overline{\delta} > 0$, we write (for $i = 1$)

$$R^\varepsilon = V_1^{5+\varepsilon} + O(V_1^{4+\varepsilon}V_2) - \overline{V}_1^5 - \overline{V}_2^5 \le C\varepsilon\overline{V}_1^5|\ln\overline{V}_1| + O(V_1^{4+\varepsilon}V_2) + O(\varepsilon^5)$$

whence

$$|\overline{V}^{-4}R^\varepsilon| \le C\varepsilon.$$

In the exterior of these two regions, we see that $|R^\varepsilon| \le C\varepsilon^5$, and since $\|\cdot\|_{**}$ is equivalent to $\|\overline{V}^{-4}\cdot\|_\infty$, the first inequality in (3.22) follows. Next, we write

$$\partial_{\xi_1'}R^\varepsilon = (5+\varepsilon)V^{4+\varepsilon}(\partial_{\xi_1'}V_1) - 5\overline{V}_1^4(\partial_{\xi_1'}\overline{V}_1). \tag{3.23}$$

Setting $V_1 = \overline{V}_1 - f_1$, we get

$$\partial_{\xi_1'}R^\varepsilon = 5(V^{4+\varepsilon} - \overline{V}_1^4)(\partial_{\xi_1'}\overline{V}_1) - 5V^{4+\varepsilon}(\partial_{\xi_1'}f_1) + \varepsilon V^{4+\varepsilon}(\partial_{\xi_1'}V_1).$$

Arguing as above, we have in the region $|x-\xi_1'| \le \overline{\delta}/\varepsilon$, for small $\overline{\delta} > 0$

$$\begin{aligned} V^{4+\varepsilon} - \overline{V}_1^4 &= V_1^{4+\varepsilon} - \overline{V}_1^4 + O(V_1^3V_2) \\ &= \overline{V}_1^{4+\varepsilon} - \overline{V}_1^4 + O(\overline{V}_1^3f_1 + V_1^3V_2) \\ &= \varepsilon\overline{V}_1^4\ln\overline{V}_1 + O(\varepsilon^2\overline{V}_1^4(\ln\overline{V}_1)^2 + \overline{V}_1^3f_1 + V_1^3V_2) \end{aligned}$$

and, using the fact that $V_2 = O(\varepsilon)$ in the considered region, $|\partial_{\xi_1'}\overline{V}_1| \le C\overline{V}_1^2$ and, through the maximum principle, $f_1 = O(\varepsilon)$ in Ω_ε, uniformly with respect to the parameters, provided that (2.4) is satisfied, we obtain

$$\overline{V}^{-4}\partial_{\xi_1'}R^\varepsilon \le C\varepsilon.$$

In the region $|x-\xi_2'| \le \overline{\delta}/\varepsilon$, for small $\overline{\delta} > 0$

$$\overline{V}^{-4}|(V^{4+\varepsilon} - \overline{V}_1^4)(\partial_{\xi_1'}\overline{V}_1)| \le C\overline{V}_1^2 \le C\varepsilon^2$$

whence again, in that region

$$\overline{V}^{-4}|\partial_{\xi_1'}R^\varepsilon| \le C\varepsilon.$$

In the exterior of the previous regions we derive from (3.23) that

$$|\partial_{\xi_1'}R^\varepsilon| \le C\overline{V}^6 \le C\varepsilon^6$$

from which we deduce

$$\|\partial_{\xi'} R^\varepsilon\|_{**} \le C\varepsilon. \tag{3.24}$$

In order to estimate $\|\partial^2_{\xi'\Lambda} R^\varepsilon\|_{**}$, we write

$$\begin{aligned}\partial^2_{\xi_1'\Lambda_1} R^\varepsilon &= (5+\varepsilon)(4+\varepsilon)(\partial_{\xi_1'} V_1)(\partial_{\Lambda_1} V_1)(V_1+V_2)^{3+\varepsilon} \\ &\quad + (5+\varepsilon)(V_1+V_2)^{4+\varepsilon}(\partial^2_{\xi_1'\Lambda_1} V_1) - 5\overline{V}_1^4(\partial^2_{\xi_1'\Lambda_1}\overline{V}_1) \\ &\quad - 20(\partial_{\xi_1'}\overline{V}_1)(\partial_{\Lambda_1}\overline{V}_1)\overline{V}_1^3.\end{aligned}$$

Away from ξ_1', noticing that $\partial_{\Lambda_1} V_1 = O(\overline{V}_1)$, $\partial^2_{\xi_1'\Lambda_1} V_1 = O(\overline{V}_1^2)$ in Ω_ε (and the same estimates hold for $\overline{V}_1$ instead of V_1)

$$|\partial^2_{\xi_1'\Lambda_1} R^\varepsilon| \le C(\varepsilon^3\overline{V}^3 + \varepsilon^2\overline{V}^4 + \varepsilon^6)$$

whence in this region, since $\overline{V} \ge C\varepsilon$ in Ω_ε

$$\overline{V}^{-4}|\partial^2_{\xi_1'\Lambda_1} R^\varepsilon| \le C\varepsilon^2.$$

Looking at the region $|x-\xi_1'| \le \overline{\delta}/\varepsilon$, for small $\overline{\delta} > 0$, we write $\partial^2_{\xi_1'\Lambda_1} R^\varepsilon$ in the following way

$$\begin{aligned}\partial^2_{\xi_1'\Lambda_1} R^\varepsilon &= 20\big[(\partial_{\xi_1'} V_1)(\partial_{\Lambda_1} V_1)(V_1+V_2)^{3+\varepsilon} - (\partial_{\xi_1'}\overline{V}_1)(\partial_{\Lambda_1}\overline{V}_1)\overline{V}_1^3\big] \\ &\quad + 5\big[(V_1+V_2)^{4+\varepsilon}(\partial^2_{\xi_1'\Lambda_1} V_1) - \overline{V}_1^4(\partial^2_{\xi_1'\Lambda_1}\overline{V}_1)\big] \\ &\quad + \varepsilon(9+\varepsilon)(\partial_{\xi_1'} V_1)(\partial_{\Lambda_1} V_1)(V_1+V_2)^{3+\varepsilon} + \varepsilon(V_1+V_2)^{4+\varepsilon}(\partial^2_{\xi_1'\Lambda_1} V_1) \\ &= I_1 + I_2 + I_3 + I_4.\end{aligned}$$

We have

$$\overline{V}^{-4}(|I_3| + |I_4|) \le C\varepsilon\overline{V}^2 \le C\varepsilon.$$

Writing $V_1 = \overline{V}_1 - f_1$, we have also

$$\begin{aligned}I_1 &= (\partial_{\xi_1'}\overline{V}_1)(\partial_{\Lambda_1}\overline{V}_1)\big[V^{3+\varepsilon} - \overline{V}_1^3\big] \\ &\quad + \big[(\partial_{\xi_1'} f_1)(\partial_{\Lambda_1} f_1) - (\partial_{\xi_1'}\overline{V}_1)(\partial_{\Lambda_1} f_1) - (\partial_{\Lambda_1}\overline{V}_1)(\partial_{\xi_1'} f_1)\big] V^{3+\varepsilon}\end{aligned}$$

and

$$(\partial_{\xi_1'}\overline{V}_1)(\partial_{\Lambda_1}\overline{V}_1)\big[V^{3+\varepsilon} - \overline{V}_1^3\big] = O\big[\varepsilon\overline{V}^4|\overline{V}_1^3 \ln \overline{V}_1| + \overline{V}^5(f_1 + V_2)\big].$$

Consequently

$$\overline{V}^{-4}|I_1| \le C\varepsilon$$

and similarly

$$\overline{V}^{-4}|I_2| \le C\varepsilon.$$

The computations for estimating $\|\partial^2_{\xi_1'\Lambda_2} R^\varepsilon\|_{**}$ are simpler and we omit them. Finally, we obtain

$$\|\partial^2_{\xi'\Lambda} R^\varepsilon\|_{**} \le C\varepsilon. \tag{3.25}$$

$\|\partial_\Lambda R^\varepsilon\|_{**}$, $\|\partial^2_{\xi'^2} R^\varepsilon\|_{**}$ and $\|\partial^2_{\Lambda^2} R^\varepsilon\|_{**}$ may be estimated in the same way, concluding the proof of the lemma. $\square$

Lemma 3.3 *The map $(\xi^{'},\Lambda)\mapsto\phi(\xi^{'},\Lambda)$ provided by Proposition 2.2 is C^2 for the norm $\|\cdot\|_*$, and there exists C independent of ε and ξ', Λ satisfying (2.4) such that*

$$\|D_{(\xi^{'},\Lambda)}\,\phi\|_*\le C\varepsilon^2 \qquad\qquad \|D^2_{(\xi^{'},\Lambda)}\,\phi\|_*\le C\varepsilon^2. \tag{3.26}$$

Proof. We recall that ϕ given by Proposition 2.2 is defined through the relation

$$\phi=-L_\varepsilon(N_\varepsilon(\phi+\psi)) \tag{3.27}$$

and

$$N_\varepsilon(\bar\phi)=N_\varepsilon(\xi^{'},\Lambda,\bar\phi)=(V+\bar\phi)_+^{5+\varepsilon}-V^{5+\varepsilon}-(5+\varepsilon)V^{4+\varepsilon}\bar\phi. \tag{3.28}$$

Setting

$$\mathcal{O}^\varepsilon_\delta=\Big\{\xi^{'}=(\xi^{'}_1,\xi^{'}_2)\in\Omega_\varepsilon\times\Omega_\varepsilon\ :|\xi^{'}_1-\xi^{'}_2|>\delta/\varepsilon,\ d(\xi^{'}_i,\partial\Omega)>\delta/\varepsilon,\ i=1,2\Big\}$$

and $\mathcal{F}=\{\tilde\phi\in H^1_0(\Omega_\varepsilon)\ :\ \|\tilde\phi\|_*\le\varepsilon\}$, we define the map $B\ :\mathcal{O}^\varepsilon_\delta\times\big(]\delta,\delta^{-1}[\big)^2\times\mathcal{F}\to L^\infty_*(\Omega_\varepsilon)$ as

$$B(\xi^{'},\Lambda,\tilde\phi)\equiv\tilde\phi+L_\varepsilon(N_\varepsilon(\tilde\phi+\psi)).$$

We have

$$\partial_{\tilde\phi}B(\xi^{'},\Lambda,\tilde\phi)[\theta]=\theta+L_\varepsilon\big((\partial_{\bar\phi}N_\varepsilon)(\xi^{'},\Lambda,\tilde\phi+\psi)\ \theta\big)\equiv\theta+M(\theta)=(I+M)(\theta)$$

and, according to Proposition 2.1

$$\|M(\theta)\|_*\le C\|(\partial_{\bar\phi}N_\varepsilon)(\xi^{'},\Lambda,\tilde\phi+\psi)\theta\|_{**}\le C\|\overline{V}^{-3}(\partial_{\bar\phi}N_\varepsilon)(\xi^{'},\Lambda,\tilde\phi+\psi)\|_\infty\|\theta\|_*.$$

(3.28) yields

$$(\partial_{\bar\phi}N_\varepsilon)(\xi^{'},\Lambda,\bar\phi)=(5+\varepsilon)[(V+\bar\phi)_+^{4+\varepsilon}-V^{4+\varepsilon}] \tag{3.29}$$

whence, using (3.21) and $\tilde\phi\in\mathcal{F}$

$$\overline{V}^{-3}|(\partial_{\bar\phi}N_\varepsilon)(\xi^{'},\Lambda,\tilde\phi+\psi)|\le C\varepsilon \tag{3.30}$$

and $\|M(\theta)\|_*\le C\varepsilon\|\theta\|_*$. As a consequence, for ε small enough, the linear operator $\partial_{\tilde\phi}B(\xi^{'},\Lambda,\tilde\phi)$ is invertible in $L^\infty_*(\Omega_\varepsilon)$, with uniformly bounded inverse. As $(\xi^{'},\Lambda,\tilde\phi)\mapsto B(\xi^{'},\Lambda,\tilde\phi)$ is C^2, $(\xi^{'},\Lambda)\mapsto\phi(\xi^{'},\Lambda)$ defined by $B(\xi^{'},\Lambda,\phi(\xi^{'},\Lambda))=0$ is also a C^2-map.

Let us prove (3.26). Derivating $B(\xi^{'},\Lambda,\phi(\xi^{'},\Lambda))=0$ with respect to $\xi^{'}$, we find

$$\begin{aligned}\partial_{\xi'}\phi\ &=[\partial_{\tilde\phi}B]^{-1}\quad[\partial_{\xi'}B]\\ &=[\partial_{\tilde\phi}B]^{-1}\quad\Big[(\partial_{\xi'}L_\varepsilon)\big(N_\varepsilon(\phi+\psi)\big)+L_\varepsilon\big((\partial_{\xi'}N_\varepsilon)(\phi+\psi)\big)\\ &\qquad\qquad\qquad+L_\varepsilon\big((\partial_{\bar\phi}N_\varepsilon)(\phi+\psi)\ \partial_{\xi'}\psi\big)\Big]\end{aligned}$$

whence, in view of Proposition 2.1

$$\|\partial_{\xi'}\phi\|_*\le C\big[\|N_\varepsilon(\phi+\psi)\|_{**}+\|(\partial_{\xi'}N_\varepsilon)(\phi+\psi)\|_{**}+\|(\partial_{\bar\phi}N_\varepsilon)(\phi+\psi)\ \partial_{\xi'}\psi\|_{**}\big].$$

From (3.28), (3.21) and (2.14) we deduce that

$$\|N_\varepsilon(\phi+\psi)\|_{**} \leq C\varepsilon^2. \tag{3.31}$$

We also compute, recalling that $|\partial_{\xi'} V| \leq C\overline{V}^2$, and assuming that $\|\bar{\phi}\|_* \leq C\varepsilon$

$$\overline{V}^{-4} |(\partial_{\xi'} N_\varepsilon)(\xi', \Lambda, \bar{\phi})| \leq C\overline{V}^{-2} \left|(V+\bar{\phi})_+^{4+\varepsilon} - V^{4+\varepsilon} - (4+\varepsilon)V^{3+\varepsilon}\bar{\phi}\right| \leq C\|\bar{\phi}\|_*^2$$

whence

$$\|(\partial_{\xi'} N_\varepsilon)(\xi', \Lambda, \phi+\psi)\|_{**} \leq C\varepsilon^2.$$

Lastly, using (3.30) and (3.21)

$$\|(\partial_{\bar{\phi}} N_\varepsilon)(\phi+\psi)\ \partial_{\xi'}\psi\|_{**} \leq C\|\overline{V}^{-3}(\partial_{\bar{\phi}} N_\varepsilon)(\phi+\psi)\|_\infty \|\partial_{\xi'}\psi\|_* \leq C\varepsilon^2$$

and finally we obtain

$$\|\partial_{\xi'}\phi\|_* \leq C\varepsilon^2.$$

The estimate for $\partial_\Lambda \phi$ is obtained in the same way. We turn now to the second derivatives. We concentrate our attention on $\partial^2_{\xi'\Lambda}\phi$, since the estimates for $\partial^2_{\Lambda^2}\phi$ and $\partial^2_{\xi'^2}\phi$ follow from identical arguments. Derivating $B(\xi', \Lambda, \phi(\xi', \Lambda)) = 0$ with respect to ξ' and Λ, we find

$$\partial^2_{\xi'\Lambda}\phi = [\partial_{\bar{\phi}} B]^{-1}\left[\partial^2_{\bar{\phi}^2} B.\partial_{\xi'}\phi.\partial_\Lambda\phi + \partial^2_{\bar{\phi}\xi'} B.\partial_\Lambda\phi + \partial^2_{\bar{\phi}\Lambda} B.\partial_{\xi'}\phi + \partial^2_{\xi'\Lambda} B\right] \tag{3.32}$$

and we have to estimate each term of the right hand side in norm $\|\cdot\|_{**}$. Let us consider the first term. According to the definition of B and (3.28)

$$\begin{aligned}\partial^2_{\bar{\phi}^2} B.\partial_{\xi'}\phi.\partial_\Lambda\phi &= L_\varepsilon\left[\partial^2_{\bar{\phi}^2} N_\varepsilon.\partial_{\xi'}\phi.\partial_\Lambda\phi\right] \\ &= L_\varepsilon\left[(5+\varepsilon)(4+\varepsilon)(V+\phi+\psi)_+^{3+\varepsilon}(\partial_{\xi'}\phi)(\partial_\Lambda\phi)\right]\end{aligned}$$

and in view of Proposition 2.2, we have to estimate

$$\|(V+\phi+\psi)_+^{3+\varepsilon}(\partial_{\xi'}\phi)(\partial_\Lambda\phi)\|_{**} \leq \|\overline{V}^{-2}(V+\phi+\psi)_+^{3+\varepsilon}\|_\infty \|\partial_{\xi'}\phi\|_* \|\partial_\Lambda\phi\|_* \leq C\varepsilon^2$$

using (3.21) and Proposition 2.2. Estimating the other terms in the same way provides us with the announced result. □

We are now able to prove the main result of this Section. From Proposition 2.3, we know that $u = \hat{U} + \hat{\psi} + \hat{\phi}$ defined through (2.16) (2.17) (2.18) is a solution to the initial problem if and only if (ξ, Λ) is a critical point of $I_\varepsilon = J_\varepsilon(\hat{U} + \hat{\psi} + \hat{\phi})$.

Proposition 3.1 *We have the expansion*

$$I_\varepsilon(\xi, \Lambda) = C_1 + C_2\varepsilon\ln\varepsilon + C_3\varepsilon + C_2\varepsilon\Psi(\xi, \Lambda) + \varepsilon\theta_\varepsilon(\xi, \Lambda) \tag{3.33}$$

with θ_ε, $D_{\xi,\Lambda}\theta_\varepsilon$ *and* $D^2_{\xi,\Lambda}\theta_\varepsilon$ *going to zero as* ε *goes to zero, uniformly with respect to* ξ, Λ *satisfying (2.4).*

Proof. We claim that

$$I_\varepsilon(\xi,\Lambda)=J_\varepsilon(\hat U)+\varepsilon\rho_\varepsilon(\xi,\Lambda) \tag{3.34}$$

with ρ_ε, $D_{(\xi,\Lambda)}\rho_\varepsilon$ and $D^2_{(\xi,\Lambda)}\rho_\varepsilon$ going to zero as ε goes to zero, uniformly with respect to ξ, Λ satisfying (2.4). Then, expansion (3.33) is obtained estimating $J_\varepsilon(\hat U)$. Indeed, we have

$$\begin{aligned}
\varepsilon^{2\zeta-1}J_\varepsilon(\hat U)&=\varepsilon^{2\zeta-1}J_\varepsilon(\varepsilon^{\frac12-\zeta}(U_1+U_2))\\
&=J_\varepsilon(U_1+U_2)+\frac{1-\varepsilon^{\frac{\varepsilon}{2}}}{6+\varepsilon}\int_\Omega(U_1+U_2)^{6+\varepsilon}\\
&=J_\varepsilon(U_1+U_2)+\frac16\Big(-\frac{\varepsilon}{2}\ln\varepsilon+o(\varepsilon)\Big)\Big(2\int_{\mathbb{R}^3}\overline{U}^{6+\varepsilon}+o(\varepsilon\ln|\varepsilon|)\Big)\\
&=J_\varepsilon(U_1+U_2)-\sqrt3\frac{\pi^2}{8}\varepsilon\ln\varepsilon+o(\varepsilon)
\end{aligned}$$

whence

$$J_\varepsilon(\hat U)=\Big(1+\frac14\varepsilon\ln\varepsilon+o(\varepsilon)\Big)\Big(J_\varepsilon(U_1+U_2)-\sqrt3\frac{\pi^2}{8}\varepsilon\ln\varepsilon+o(\varepsilon)\Big)$$

and the desired expansion for $J_\varepsilon(\hat U)$ using Lemma 2.1 and (2.3). The expansions for the derivatives of $J_\varepsilon(\hat U)$ are obtained in the same way.

We turn now to the proof of claim (3.34). Validity of such an expansion for I_ε and its first derivatives has already been proved in [5] (Proposition 6.1). Moreover, it can easily be found again from the arguments used below, concerning the second derivatives. Actually, we shall concentrate our attention on the second derivative with respect to ξ, Λ, for the second derivatives with respect to Λ^2 and ξ^2 may be treated exactly in the same way. The proof is composed of two steps : we show that

$$\partial^2_{\xi\Lambda}[I(\xi,\Lambda)-J_\varepsilon(\hat U+\hat\psi)]=o(\varepsilon) \tag{3.35}$$

and

$$\partial^2_{\xi\Lambda}[J_\varepsilon(\hat U+\hat\psi)-J_\varepsilon(\hat U)]=o(\varepsilon). \tag{3.36}$$

Let us prove (3.35). From a Taylor expansion and the fact that $J'_\varepsilon(\hat U+\hat\psi+\hat\phi)[\phi]=0$, we have

$$\begin{aligned}
I(\xi,\Lambda)-J_\varepsilon(\hat U+\hat\psi)&=J_\varepsilon(\hat U+\hat\psi+\hat\phi)-J_\varepsilon(\hat U+\hat\psi)=\int_0^1 D^2J_\varepsilon(\hat U+\hat\psi+t\hat\phi)[\hat\phi,\hat\phi]tdt\\
&=\varepsilon^{1-2\zeta}\int_0^1\Big(\int_{\Omega_\varepsilon}N_\varepsilon(\phi+\psi)\phi+\int_{\Omega_\varepsilon}(5+\varepsilon)\big[V^{4+\varepsilon}-(V+\psi+t\phi)_+^{4+\varepsilon}\big]\phi^2\Big)\,tdt.
\end{aligned} \tag{3.37}$$

We recall that $\zeta=(2+\frac12\varepsilon)^{-1}<1/2$. Differentiating twice with respect to ξ,Λ, we find

$$\begin{aligned}
\partial^2_{\xi\Lambda}[I(\xi,\Lambda)-J_\varepsilon(\hat U+\hat\psi)]=\varepsilon^{-2\zeta}\int_0^1\Big(&\int_{\Omega_\varepsilon}\partial^2_{\xi'\Lambda}[N_\varepsilon(\phi+\psi)\phi]\\
&-\int_{\Omega_\varepsilon}(5+\varepsilon)\partial^2_{\xi'\Lambda}\big[\big((V+\psi+t\phi)_+^{4+\varepsilon}-V^{4+\varepsilon}\big)\phi^2\big]\Big)tdt
\end{aligned} \tag{3.38}$$

with $\xi_i^{'} = \xi_i/\varepsilon$. Let us estimate the last integral. Setting

$$\varphi = \psi + t\phi \qquad H = H_\varepsilon(\xi^{'}, \Lambda) = (V+\varphi)_+^{4+\varepsilon} - V^{4+\varepsilon}$$

we have

$$\begin{aligned}\partial^2_{\xi'\Lambda}(H\phi^2) =& 2(\partial_{\xi'}\phi)(\partial_\Lambda\phi)H + 2\phi(\partial^2_{\xi'\Lambda}\phi)H + 2\phi(\partial_\Lambda\phi)(\partial_{\xi'}H)\\ &+ 2\phi(\partial_\Lambda H)(\partial_{\xi'}\phi) + \phi^2(\partial^2_{\xi'\Lambda}H)\end{aligned}$$

with

$$\partial_{\xi'}H = (4+\varepsilon)[(V+\varphi)_+^{3+\varepsilon} - V^{3+\varepsilon}](\partial_{\xi'}V) + (4+\varepsilon)(V+\varphi)_+^{3+\varepsilon}(\partial_{\xi'}\varphi)$$

and

$$\begin{aligned}\partial^2_{\xi'\Lambda}H =& (4+\varepsilon)(3+\varepsilon)[(V+\varphi)_+^{2+\varepsilon} - V^{2+\varepsilon}](\partial_{\xi'}V)(\partial_\Lambda V)\\ &+ (4+\varepsilon)[(V+\varphi)_+^{3+\varepsilon} - V^{3+\varepsilon}](\partial^2_{\xi'\Lambda}V)\\ &+ (4+\varepsilon)(3+\varepsilon)(V+\varphi)_+^{2+\varepsilon}(\partial_\Lambda\varphi)[(\partial_{\xi'}V) + (\partial_{\xi'}\varphi)]\\ &+ (4+\varepsilon)(3+\varepsilon)(V+\varphi)_+^{2+\varepsilon}(\partial_{\xi'}\varphi)(\partial_\Lambda V) + (4+\varepsilon)(V+\varphi)_+^{3+\varepsilon}(\partial^2_{\xi'\Lambda}\varphi).\end{aligned}$$

One the one hand, we deduce from the definition of V

$$|\partial_\Lambda V| \le C\overline{V} \qquad |\partial_{\xi'}V| \le C\overline{V}^2 \qquad |\partial^2_{\xi'\Lambda}| \le C\overline{V}^2 \tag{3.39}$$

uniformly in Ω_ε, with C independent of ε and $\xi^{'}$, Λ satisfying (2.4). One the other hand, (3.21) and (3.26) provide us with estimates of ψ and ϕ in norm $\|\cdot\|_*$. Lastly, we notice that for any $\gamma > 3$

$$\int_{\Omega_\varepsilon} \overline{V}^\gamma = O(1) \tag{3.40}$$

as ε goes to zero, uniformly with respect to $\xi^{'}$, Λ satisfying (2.4). Using these informations, straightforward computations yield

$$\varepsilon^{-2\zeta}\int_{\Omega_\varepsilon} \partial^2_{\xi'\Lambda}(H\phi^2) = o(\varepsilon).$$

The same arguments applied to the first integral in (3.38) lead to a similar estimate, establishing (3.35). We turn now to the proof of (3.36). A Taylor expansion gives

$$\begin{aligned}J_\varepsilon(\hat{U}+\hat{\psi}) - J_\varepsilon(\hat{U}) =& \varepsilon^{1-2\zeta}\big(\mathcal{I}_\varepsilon(V+\psi) - \mathcal{I}_\varepsilon(V)\big)\\ =& \varepsilon^{1-2\zeta}\Big(D\mathcal{I}_\varepsilon(V)[\psi] + \int_0^1 (1-t)D^2\mathcal{I}_\varepsilon(V+t\psi)[\psi,\psi]\Big).\end{aligned}$$

As $D\mathcal{I}_\varepsilon(V)[\psi] = -\int_{\Omega_\varepsilon}(\Delta V + V^{5+\varepsilon})\psi$ and $\Delta V = \Delta(V_1+V_2) = -\overline{V}_1^5 - \overline{V}_2^5 = R^\varepsilon - V^{5+\varepsilon}$, we obtain

$$D\mathcal{I}_\varepsilon(V)[\psi] = -\int_{\Omega_\varepsilon} R^\varepsilon\psi.$$

On the other hand

$$D^2\mathcal{I}_\varepsilon(V+t\psi)[\psi,\psi]=\int_{\Omega_\varepsilon}|\nabla\psi|^2-(5+\varepsilon)\int_{\Omega_\varepsilon}(V+t\psi)_+^{4+\varepsilon}\psi^2.$$

Then, integration by parts and $\psi=-L_\varepsilon(R_\varepsilon)$ yield

$$D^2\mathcal{I}_\varepsilon(V+t\psi)[\psi,\psi]=\int_{\Omega_\varepsilon}R^\varepsilon\psi-(5+\varepsilon)\int_{\Omega_\varepsilon}\big((V+t\psi)_+^{4+\varepsilon}-V^{4+\varepsilon}\big)\psi^2.$$

Consequently

$$\begin{aligned}&J_\varepsilon(\hat U+\hat\psi)-J_\varepsilon(\hat U)\\&\quad=\varepsilon^{1-2\zeta}\left(-\frac12\int_{\Omega_\varepsilon}R^\varepsilon\psi-(5+\varepsilon)\int_0^1(1-t)(\int_{\Omega_\varepsilon}\big[(V+t\psi)_+^{4+\varepsilon}-V^{4+\varepsilon}\big]\psi^2)dt\right)\end{aligned}$$

and

$$\begin{aligned}&\partial^2_{\xi\Lambda}\big[J_\varepsilon(\hat U+\hat\psi)-J_\varepsilon(\hat U)\big]\\&\quad=\varepsilon^{-2\zeta}\left(-\frac12\partial^2_{\xi'\Lambda}(\int_{\Omega_\varepsilon}R^\varepsilon\psi)-(5+\varepsilon)\int_0^1(1-t)(\int_{\Omega_\varepsilon}\partial^2_{\xi'\Lambda}\big[\big((V+t\psi)_+^{4+\varepsilon}-V^{4+\varepsilon}\big)\psi^2\big])dt\right).\end{aligned}$$

We first consider the last integral. Denoting by

$$K=K_\varepsilon(\xi',\Lambda)(x)=(V+t\psi)_+^{4+\varepsilon}-V^{4+\varepsilon}$$

we have

$$\begin{aligned}\partial^2_{\xi'\Lambda}(K\psi^2)=&2(\partial_{\xi'}\psi)(\partial_\Lambda\psi)K+2\psi(\partial^2_{\xi'\Lambda}\psi)K+2\psi(\partial_\Lambda\psi)(\partial_{\xi'}K)\\&+2\psi(\partial_\Lambda K)(\partial_{\xi'}\psi)+\psi^2(\partial^2_{\xi'\Lambda}K)\end{aligned}$$

with

$$\partial_{\xi'}K=(4+\varepsilon)[(V+t\psi)_+^{3+\varepsilon}-V^{3+\varepsilon}](\partial_{\xi'}V)+(4+\varepsilon)t(V+t\psi)_+^{3+\varepsilon}(\partial_{\xi'}\psi).$$

A similar expression holds for $\partial_\Lambda K$, and

$$\begin{aligned}\partial^2_{\xi'\Lambda}K=&(4+\varepsilon)(3+\varepsilon)[(V+t\psi)_+^{2+\varepsilon}-V^{2+\varepsilon}](\partial_{\xi'}V)(\partial_\Lambda V)\\&+(4+\varepsilon)(3+\varepsilon)(V+t\psi)_+^{2+\varepsilon}t(\partial_\Lambda\psi)(\partial_{\xi'}V)\\&+(4+\varepsilon)[(V+t\psi)_+^{3+\varepsilon}-V^{3+\varepsilon}](\partial^2_{\xi'\Lambda}V)\\&+(4+\varepsilon)(3+\varepsilon)t(V+t\psi)_+^{2+\varepsilon}(\partial_{\xi'}\psi)(\partial_\Lambda V)\\&+(4+\varepsilon)(3+\varepsilon)t^2(V+t\psi)_+^{2+\varepsilon}(\partial_{\xi'}\psi)(\partial_\Lambda\psi)\\&+(4+\varepsilon)t(V+t\psi)_+^{3+\varepsilon}(\partial^2_{\xi'\Lambda}\psi).\end{aligned}$$

Once again, the estimate

$$\varepsilon^{-2\zeta}\left|\int_{\Omega_\varepsilon}\psi^2(\partial^2_{\xi'\Lambda}K)\right|=o(\varepsilon)$$

follows directly from (3.21), (3.26), (3.39) and (3.40).

In order to complete the proof of (3.36), it only remains to estimate the quantity

$$\varepsilon^{-2\zeta}\,\partial^2_{\xi'\Lambda}\Big(\int_{\Omega_\varepsilon} R^\varepsilon\psi\Big) = \varepsilon^{-2\zeta}\int_{\Omega_\varepsilon} R^\varepsilon(\partial^2_{\xi'\Lambda}\psi) + (\partial_\Lambda\psi)(\partial_{\xi'}R^\varepsilon) + (\partial_{\xi'}\psi)(\partial_\Lambda R^\varepsilon) + \psi(\partial^2_{\xi'\Lambda}R^\varepsilon). \tag{3.41}$$

Among the four integral terms which occur on the right hand side of (3.41), let us consider, for example, the first one

$$I_\varepsilon = \varepsilon^{-2\zeta}\int_{\Omega_\varepsilon} R^\varepsilon(\partial^2_{\xi'\Lambda}\psi).$$

Let $R>0$, and $\Omega'_\varepsilon = \Omega_\varepsilon\setminus\big(B(\xi'_1,R)\cup B(\xi'_2,R)\big)$. We have

$$\begin{aligned}\varepsilon^{-2\zeta}\int_{\Omega'_\varepsilon} R^\varepsilon(\partial^2_{\xi'\Lambda}\psi) &= O\Big(\varepsilon^{-1}\|\partial^2_{\xi'\Lambda}\psi\|_*\|R^\varepsilon\|_{**}\int_{\Omega'_\varepsilon}\overline{V}^5\Big)\\ &= O\Big(\varepsilon\int_{\Omega'_\varepsilon}\overline{V}^5\Big)\end{aligned}$$

using (3.22) and (3.21), and

$$\int_{\Omega'_\varepsilon}\overline{V}^5 \le C\int_R^{+\infty}\frac{r^2}{(1+r^2)^2}dr \le \frac{C'}{R}$$

with C' a constant independent of R. Therefore

$$\varepsilon^{-2\zeta}\int_{\Omega'_\varepsilon} R^\varepsilon(\partial^2_{\xi'\Lambda}\psi) = O\big(\frac{\varepsilon}{R}\big). \tag{3.42}$$

On the other hand, according to the definition (2.8) of R^ε there exists a constant c_i such that

$$\varepsilon^{-1}R^\varepsilon(\xi'_i + x) \longrightarrow \overline{V}^5_{0,\Lambda^*_i}\ln\overline{V}_{0,\Lambda^*_i} + c_i\overline{V}^4_{0,\Lambda^*_i}$$

uniformly on $B(\xi'_i,R)$ (with $\Lambda^*_i = (c_N\Lambda_i^2)^{\frac{1}{N-2}}$). We recall that $\psi = -L_\varepsilon(R^\varepsilon)$ is the unique solution of

$$\begin{cases}\Delta\psi + (5+\varepsilon)V^{4+\varepsilon}\psi &= -R^\varepsilon + \sum_{i,j}c_{ij}V_i^4Z_{ij} \quad \text{in } \Omega_\varepsilon\\ \psi &= 0 \quad \text{on } \partial\Omega_\varepsilon\\ < V_i^4Z_{ij},\psi > &= 0 \quad \text{for all } i,j\end{cases}$$

for some numbers c_{ij}, with $|c_{ij}|\le C\|R^\varepsilon\|_{**}\le C'\varepsilon$. Then, proceeding as in [15] (proof of Proposition 3.3), we obtain that, up to a subsequence, $\varepsilon^{-1}\psi$ converges uniformly in $B(\xi'_i,R)$, as ε goes to 0, to a solution Φ_i of

$$\begin{cases}\Delta\Phi + 5\overline{V}^4_{0,\Lambda^*_i}\Phi &= -\overline{V}^5_{0,\Lambda^*_i}\ln\overline{V}_{0,\Lambda^*_i} - c_i\overline{V}^4_{0,\Lambda^*_i} + \sum_{i,j}d_{ij}\overline{V}^4_{0,\Lambda^*_i}\overline{Z}_{ij} \quad \text{in } \mathbb{R}^3\\ < \overline{V}^4_{0,\Lambda^*_i}\overline{Z}_{ij},\Phi > &= 0 \quad 1\le j\le 4\\ |\Phi| &\le C\overline{V}_{0,\Lambda^*_i}\end{cases} \tag{3.43}$$

for some number d_{ij}. Multiplying the equation by $\overline{Z}_{ij}, 1\le j\le 3$, and integrating in $\mathbb{R}^3$, the orthogonality relations satisfied by Φ and oddness of $\overline{Z}_{ij}$ with respect to the variable $(x-\xi'_i)_j$

yield $d_{ij} = 0, 1 \leq j \leq 3$. On the other hand, the kernel of the operator $L_i = \Delta + 5\overline{V}^4_{0,\Lambda_i^*}$ in $W^{2,r}(\mathbb{R}^3)$, $3 < r < +\infty$, is spanned by the $\overline{Z}_{ij}$'s, $1 \leq j \leq 4$ (Lemma 2.3 in [15]). $|\Phi_i| \leq C\overline{V}_{0,\Lambda_i^*}$ implies that $\Phi_i \in W^{2,r}(\mathbb{R}^3)$ for r large enough. Then, if Φ_i and $\Phi_i^{'}$ are two solutions of (3.43), $\Phi_i - \Phi_i^{'} \in Ker\ L_i$, and the orthogonality relations $< \overline{V}^4_{0,\Lambda_i^*}\overline{Z}_{ij}, \Phi_i - \Phi_i^{'} >= 0$, $1 \leq j \leq 4$, imply that $\Phi_i = \Phi_i^{'}$. Since the solution Φ_i of (3.43) is unique, it is radial (otherwise, a rotation of Φ_i would provide us with another solution). Proceeding in the same way, we would prove that, up to a subsequence

$$\varepsilon^{-1}(\partial^2_{\xi'\Lambda}\psi) \longrightarrow \partial^2_{\xi'\Lambda}\Phi_i \quad \text{uniformly in} \quad B(\xi_i^{'}, R).$$

$\partial^2_{\xi'\Lambda}\Phi_i$ being odd in the variables $(x - \xi_i^{'})_j$, we finally see that

$$\varepsilon^{-1}\int_{B(\xi_i^{'}, R)} R^\varepsilon(\partial^2_{\xi'\Lambda}\psi)\, dx = o(\varepsilon) \quad \text{as } \varepsilon \text{ goes to } 0.$$

Since R may be chosen as large as desired, this result, together with (3.42), shows that $I_\varepsilon = o(\varepsilon)$. The other terms in (3.41) may be treated in the same way, completing the proof of (3.36).

□

4 Proof of Theorem 1.1

According to the statement of Theorem 1.1, we assume in this section that there exists a and b, $a < b < 0$, such that $H_*(\varphi^b, \varphi^a)$ is nontrivial. In order to prove Theorem 1.1, we have to show that for ε small enough, (P_ε) has a solution, which blows up at two points ξ_1, ξ_2 as ε goes to zero, with $a \leq \varphi(\xi_1, \xi_2) \leq b$ and $\nabla\varphi(\xi_1, \xi_2) = 0$.

In view of Proposition 2.3 and Proposition 3.1, we have to prove the existence of a critical point of

$$I_\varepsilon(\xi, \Lambda) = C_1 + C_3\varepsilon + C_2\varepsilon\Psi(\xi, \Lambda) + \varepsilon\theta_\varepsilon(\xi, \Lambda) \tag{4.1}$$

with $\theta_\varepsilon = o(1)$, $D_{(\xi,\Lambda)}\theta_\varepsilon = o(1)$, $D^2_{(\xi,\Lambda)}\theta_\varepsilon = o(1)$ as ε goes to 0, uniformly with respect to $(\xi, \Lambda) \in \mathcal{O}_\delta(\Omega)\times]\delta, \delta^{-1}[^2$, and Ψ is given by (2.2).

First, we remark that for any $\xi \in \Omega^2$, such that $\varphi(\xi) < 0$, $\Lambda \longmapsto \Psi(\xi, \Lambda)$ has a unique critical point $\bar{\Lambda}(\xi)$ in $(\mathbb{R}_+^*)^2$, such that

$$\bar{\Lambda}_i^2(\xi) = -\frac{H(\xi_j, \xi_j)^{1/2}}{H(\xi_i, \xi_i)^{1/2}}\,\frac{1}{\varphi(\xi)} \quad i, j = 1, 2 \ \ i \neq j. \tag{4.2}$$

Note that $\xi \in \mathcal{O}_{\delta_1}$ and $\varphi(\xi) < -\delta_2$, with δ_1, δ_2 strictly positive constants, imply the existence of $\delta_3 > 0$ such that $\delta_3 < \bar{\Lambda}_i(\xi) < \delta_3^{-1}$, $i = 1, 2$. Note also that

$$\begin{vmatrix} (\partial^2_{\Lambda_1^2}\Psi) & (\partial^2_{\Lambda_1\Lambda_2}\Psi) \\ (\partial^2_{\Lambda_1\Lambda_2}\Psi) & (\partial^2_{\Lambda_2^2}\Psi) \end{vmatrix} \big(\xi, \bar{\Lambda}(\xi)\big) = 4H(\xi_1, \xi_1)^{1/2}H(\xi_2, \xi_2)^{1/2}\varphi(\xi) < -\delta_4 < 0$$

for some $\delta_4 > 0$. Therefore, the implicit functions theorem provides us, for $\xi \in \mathcal{O}_{\delta_1}$, $\varphi(\xi) < -\delta_2$ and ε small enough, with the existence of $\Lambda(\xi)$ close to $\bar{\Lambda}(\xi)$ in C^1-norm as ε goes to 0, such that

$$\partial_\Lambda I_\varepsilon\big(\xi, \Lambda(\xi)\big) = 0.$$

Then, in view of (4.1), finding a critical point of $(\xi,\Lambda)\longmapsto I_\varepsilon(\xi,\Lambda)$ reduces to finding a critical point of $\xi\longmapsto \tilde I_\varepsilon(\xi)$, with

$$\tilde I_\varepsilon(\xi)=\Psi\big(\xi,\Lambda(\xi)\big)+\tilde\theta_\varepsilon(\xi)$$

and $\tilde\theta_\varepsilon=o(1),\ \nabla\tilde\theta_\varepsilon=o(1)$ as ε goes to 0, uniformly with respect to $\xi\in\mathcal{O}_{\delta_1}(\Omega)$, $\varphi(\xi)<-\delta_2$. $\Lambda(\xi)$ being C^1-close to $\overline{\Lambda}(\xi)$, (4.2) yields

$$\tilde I_\varepsilon(\xi)=-2\ln\big(-\varphi(\xi)\big)-1+\tilde{\tilde\theta}_\varepsilon(\xi) \tag{4.3}$$

with $\tilde{\tilde\theta}_\varepsilon=o(1),\ \nabla\tilde{\tilde\theta}_\varepsilon=o(1)$ as ε goes to 0, uniformly with respect to $\xi\in\mathcal{O}_{\delta_1}(\Omega)$, $\varphi(\xi)<-\delta_2$.

Arguing by contradiction, we assume that for any $\delta_1>0$, $\tilde I_\varepsilon$ has no critical point in the subset $\xi\in\mathcal{O}_{\delta_1}(\Omega)$, $a<\varphi(\xi)<b$. We note that φ has isolated critical values. This property follows from the analyticity of φ on $\Omega^2\setminus\Delta$, and the fact that (ξ^n) being a sequence of $\Omega^2\setminus\Delta$ such that $\xi^n\to\Delta$, $\varphi'(\xi^n)=0$, $\varphi(\xi^n)$ cannot be bounded (see the proof of Corollary 1.1 below). Then, assuming that b is not a critical value of φ, φ has no critical value c in $(a,a+\eta]\cup(b-\eta,b]$ for some $\eta>0$ sufficiently small. Consequently, φ^b retracts by deformation onto $\varphi^{b-\eta}$, $\varphi^{a+\eta}$ retracts by deformation onto φ^a, and $H_*\big(\varphi^{b-\eta},\varphi^{a+\eta}\big)\neq 0$ (on the boundary of Ω^2, $-\nabla\varphi$ points inward, see Lemma 4.1 below).

We are going to use the gradient of $\tilde I_\varepsilon$ to build a continuous deformation of $\varphi^{b-\eta}$ onto $\varphi^{a+\eta}$, whence a contradiction. As $\tilde I_\varepsilon$ is not defined on whole φ^b, we shall use the gradient of φ in the complementary regions.

We notice that $a<0$ and $\delta_0>0$ being given, $\varphi(\xi)>a$ and $d(\xi_i,\partial\Omega)>\delta_0,\ i=1,2$, imply that $|\xi_1-\xi_2|>\delta_0'$, with δ_0' a strictly positive constant.

For $\delta_0>0$ small enough and $d(x,\partial\Omega)\le 2\delta_0$, we denote by n_x the outward normal to $\partial\Omega$ at x', with $|x-x'|=\operatorname*{Min}_{y\in\partial\Omega}|x-y|$. We have the following lemma:

Lemma 4.1 *Let $a<b<0$, $\xi\in\Omega^2$ such that $a\le\varphi(\xi)\le b$ and $d(\xi_i,\partial\Omega)=\operatorname*{Min}_{j=1,2}d(\xi_j,\partial\Omega)\le 2\delta_0$. Then, for $\delta_0>0$ small enough, we have*

$$\partial_{\xi_i}\varphi(\xi).n_{\xi_i}>0.$$

Before proving this lemma, let us complete the proof of Theorem 1.1.

Proof of Theorem 1.1 completed. We consider $\zeta\in C^\infty(\overline\Omega\times\overline\Omega)$, $0\le\zeta\le 1$, such that

$$\zeta(\xi)=1\quad\text{if}\quad d(\xi_i,\partial\Omega)\ge 2\delta_0\quad i=1,2$$

$$\zeta(\xi)=0\quad\text{if}\quad d(\xi_i,\partial\Omega)\le \delta_0\quad i=1\text{ or }2.$$

We set

$$F=\zeta\nabla\tilde I_\varepsilon+(1-\zeta)\nabla\varphi$$

and we consider the differential flow

$$\frac{d}{dt}\xi(t)=-F\big(\xi(t)\big),\quad \xi(0)=\xi_0,\quad a\le\varphi(\xi_0)<-\delta_2. \tag{4.4}$$

According to the assumption on $\tilde I_\varepsilon$, (4.3) and Lemma 4.1, $F(\xi)$ does not vanish for $a\le\varphi(\xi_0)<-\delta_2$. On one hand, if $d(\xi_i,\partial\Omega)\le 2\delta_0,\ i=1$ or 2, (4.3) yields

$$\frac{d}{dt}\varphi(\xi)=2\zeta(\xi)\,\frac{|\nabla\varphi(\xi)|^2}{\varphi(\xi)}-\big(1-\zeta(\xi)\big)|\nabla\varphi(\xi)|^2+o_\varepsilon(1)<\eta_0<0$$

provided that ε is small enough, as Lemma 4.1 shows. Lemma 4.1 also shows that the orbits do not meet the boundary of Ω^2. On the other hand, if $d(\xi_i, \partial\Omega) \geq 2\delta_0,\ i = 1, 2$, and $a < \varphi(\xi) \leq b$, we have

$$\frac{d}{dt}\tilde{I}_\varepsilon(\xi) = -|\nabla \tilde{I}_\varepsilon(\xi)|^2 < 0. \tag{4.5}$$

From (4.3) we deduce that if $\xi_0 \in \varphi^{b-\eta}$, the orbit $\xi(t)$ with ξ_0 as initial datum satisfies $\varphi\big(\xi(t)\big) \in \varphi^b$ for any t, provided that ε is small enough. Therefore, (4.5) is valid along the orbit, and (4.3) proves that for ε sufficiently small, there is some t such that $\varphi\big(\xi(t)\big) = a + \eta$. Finally, composing the flow with a retraction of φ^b onto $\varphi^{b-\eta}$, we obtain a continuous deformation of $\varphi^{b-\eta}$ onto $\varphi^{a+\eta}$, a contradiction with $H_*(\varphi^{b-\eta}, \varphi^{a+\eta}) \neq 0$. $\square$

Actually, it is to be noticed that the previous arguments provide us, for ε small enough, with a nontrivial solution u_ε of $-\Delta u = u_+^{5+\varepsilon}$ in Ω, $u = 0$ on $\partial\Omega$. Then, the strong maximum principle ensures that $u_\varepsilon > 0$ in Ω. The concentration, as ε goes to zero, up to a subsequence, of u_ε at two points ξ_1, ξ_2 such that $a < \varphi(\xi_1, \xi_2) < b$ and $\nabla\varphi(\xi_1, \xi_2) = 0$ is a consequence of the construction of u_ε and (4.3).

Proof of Lemma 4.1. We prove the result for any dimension $N \geq 3$. From [11], we know the uniform expansion with respect to $y \in \Omega$

$$H(x, y) = \frac{b_N}{|x - y + 2d_x n_x|^{N-2}} + o\left(\frac{1}{d_x^{N-2}}\right)$$

as $d_x = d(x, \partial\Omega)$ goes to zero. In particular

$$H(x, x) = \frac{b_N}{2^{N-2} d_x^{N-2}} + o\left(\frac{1}{d_x^{N-2}}\right). \tag{4.6}$$

Assume that $a \leq \varphi(x, y) \leq b$ and d_x goes to 0. Then $|x - y|$ has also to go to 0, and we have the expansion

$$\varphi(x, y) = b_N\left(\frac{1}{2^{N-2} d_x^{\frac{N-2}{2}} d_y^{\frac{N-2}{2}}} - \frac{1}{|x - y|^{N-2}} + \frac{1}{|x - y + 2d_x n_x|^{N-2}}\right) + o\left(\frac{1}{d_x^{\frac{N-2}{2}} d_y^{\frac{N-2}{2}}}\right). \tag{4.7}$$

This expansion shows that d_x, d_y and $|x - y|$ are of the same order as these quantities go to 0. Then [11] provides us with the expansions

$$\frac{\partial H}{\partial n_x}(x, x) = \frac{(N-2)b_N}{2^{N-1} d_x^{N-1}} + o\left(\frac{1}{d_x^{N-1}}\right)$$

$$\frac{\partial G}{\partial n_x}(x, y) = -(N-1)b_N\left(\frac{d_y - d_x}{|x - y|^N} + \frac{d_x + d_y}{\big(|x - y|^2 + 4d_x d_y\big)^{\frac{N}{2}}}\right) + o\left(\frac{1}{d_x^{N-1}}\right)$$

from which we deduce

$$\frac{\partial \varphi}{\partial n_x}(x, y) = (N-2)b_N\left(\frac{1}{2^{N-1} d_x^{\frac{N}{2}} d_y^{\frac{N-2}{2}}} + \frac{d_y - d_x}{|x - y|^N} + \frac{d_x + d_y}{\big(|x - y|^2 + 4d_x d_y\big)^{\frac{N}{2}}}\right) + o\left(\frac{1}{d_x^{N-1}}\right).$$

This last quantity is clearly strictly positive as $d_x \leq d_y$ and d_x goes to 0. $\square$

Proof of Corollary 1.1. We have to prove that for a small enough, φ^a retracts by deformation on Δ. We are going to show that

$$\nabla\varphi(x,y).(y-x,x-y)<0 \qquad \text{when } \ \varphi(x,y)<a, \ \text{ with } \ a \ \text{ small enough.} \tag{4.8}$$

According to the definition of φ, we have :

$$\varphi(x,y)=H(x,x)^{1/2}H(y,y)^{1/2}+H(x,y)-\frac{b_N}{|x-y|^{N-2}} \tag{4.9}$$

$$\frac{\partial\varphi}{\partial x}(x,y)=\frac{H(y,y)^{1/2}}{H(x,x)^{1/2}}\frac{\partial H}{\partial x}(x,x)+\frac{\partial H}{\partial x}(x,y)+(N-2)b_N\frac{x-y}{|x-y|^N} \tag{4.10}$$

whence

$$\begin{aligned}\nabla\varphi(x,y).(y-x,x-y)=\bigg(&\frac{H(y,y)^{1/2}}{H(x,x)^{1/2}}\ \frac{\partial H}{\partial x}(x,x)-\frac{H(x,x)^{1/2}}{H(y,y)^{1/2}}\ \frac{\partial H}{\partial y}(y,y)\\ &+\frac{\partial H}{\partial x}(x,y)-\frac{\partial H}{\partial y}(x,y)\bigg).(y-x)-\frac{2(N-2)b_N}{|x-y|^{N-2}}.\end{aligned} \tag{4.11}$$

If x and y remain far from the boundary, say $d(x,\partial\Omega)$, $d(y,\partial\Omega)>\delta>0$, H and its derivatives remain bounded, (4.9) implies that $\frac{b_N}{|x-y|^{N-2}}>-a$, and (4.11) yields

$$\nabla\varphi(x,y).(y-x,x-y)<2(N-2)\,a+C_\delta<0 \quad \text{for } \ a \ \text{ small enough.}$$

If x and y approach the boundary, (4.9) implies that $|x-y|$ goes to 0, (4.7) that d_x and d_y are of the same order and $|x-y|/d_x$ is bounded. Moreover, [11] provides us with the estimate

$$\frac{\partial H}{\partial x}(x,y)=-(N-2)b_N\frac{x-y-2\,n_x.(x-y)\,n_x-2d_x\,n_x}{|x-y+2d_x\,n_x|^N}+o\bigg(\frac{1}{d_x^{N-2}}\bigg) \quad \text{as } \ d_x\to 0$$

and a similar expansion holds for $\frac{\partial H}{\partial y}(x,y)$. Therefore, taking account of (4.6) and (4.10), we find

$$\begin{aligned}\frac{\partial\varphi}{\partial x}(x,y)=(N-2)b_N\bigg(&\frac{n_x}{2^{N-1}d_x^{\frac{N}{2}}\,d_y^{\frac{N-2}{2}}}+\frac{x-y}{|x-y|^N}+\frac{x-y-2n_x.(x-y)n_x-2d_xn_x}{|x-y+2d_xn_x|^N}\bigg)\\ &+o\bigg(\frac{1}{d_x^{N-1}}\bigg) \quad \text{as } \ d_x\to 0.\end{aligned}$$

The same expression holds for the derivative of φ with respect to y, interchanging the roles of x and y. Note that we can write

$$y=x-(d_y-d_x)\,n_x+\tau+o(d_x)$$

with $\tau.n_x=0$, $|\tau|=O(|x-y|)$. Noticing that $n_y=n_x+o(1)$, we obtain

$$\begin{aligned}&\nabla\varphi(x,y).(y-x,x-y)\\ &\quad=2(N-2)b_N\bigg(-\frac{1}{|x-y|^{N-2}}-\frac{(d_x-d_y)^2}{2^N d_x^{\frac{N}{2}}\,d_y^{\frac{N}{2}}}+\frac{(d_x-d_y)^2-|x-y|^2}{\big[(d_x+d_y)^2+|\tau|^2\big]^{\frac{N}{2}}}\bigg)+o\bigg(\frac{1}{d_x^{N-2}}\bigg)<0\end{aligned}$$

as d_x goes to 0, $|x-y|/d_x$ bounded, since $\frac{1}{2^N d_x^{\frac{N}{2}}\,d_y^{\frac{N}{2}}}\geq\frac{1}{(d_x+d_y)^N}$. □

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